The Reconstruction of Cycle-free Partial Orders from their Abstract Automorphism Groups II : Cone Transitive CFPOs

Robert Barham

Institut für Algebra, TU Dresden

robert.barham@yahoo.co.uk

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Abstract

In this triple of papers, we examine when two cycle-free partial orders can share an abstract automorphism group. This question was posed by M. Rubin in his memoir concerning the reconstruction of trees.

In this middle paper, we adapt a method used by Shelah in [3] and [2], and by Shelah and Truss in [4] to define a cone transitive CFPO inside its automorphism group using the language of group theory.

1 Introduction

This paper draws on the methods employed in [4], which is about reconstructing the quotients of symmetric groups as permutations groups from the quotients of symmetric groups as abstract groups. This paper uses A_5, the alternating group on five elements, chosen because it's the smallest non-abelian simple group, to represent the set being permuted. This paper also uses A_5 to represent the CFPO.

We take the abstract automorphism group of a cone transitive CFPO and define the original CFPO. Section 2 is devoted to properly defining the CFPOs where we apply this method. Section 3 produces a long chain of first order formulae, starting with the 60-ary formula that states 'these automorphisms form a subgroup isomorphic to A_5, the alternating group on five elements'. There then follows a series of formulas with the goals of: defining subgroups whose support is exactly some of the extended cones of a single point; and expressing when two of these subgroups have disjoint support. These two goals are, by far, the hardest part of this paper. Afterwards, we have the relatively simple task of representing the points of the CFPO with these subgroups, and recovering the betweenness relation.

The final section examines how we can recover the order from the betweenness relation. In some circumstances, the order relation is first-order definable from the betweenness relation, but not always, and certainly not with the same formula in all circumstances. To over come this, we end this paper by giving an $L_{\omega_1,\omega}$-formula that always defines the order.

2 Cone Transitive

Definition 2.1. *Let M be a CFPO and let $x, y, z \in M$ Such that $z < x < y$. The **upwards cone** of x that contains y is the set $\{t \in M : x \notin \mathrm{Path}\langle t, y \rangle\}$. The **downwards cone** of x that contains z is the set $\{t \in M : x \notin \mathrm{Path}\langle t, z \rangle\}$.*

Definition 2.2. *We say that CFPO M is **1-transitive** if for all $x, y \in M$ there exists a $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(x) = y$.*

Definition 2.3. *We say that a CFPO M is **cone transitive** if it is 1-transitive and if C and D are both upwards cones or both downwards cones of $x \in M$ then there exists a $\varphi \in \mathrm{Aut}(M)$ such that $\varphi(C) = D$.*

Since every element may be sent to any other, if M is 1-transitive then M is monochromatic. M is cone transitive implies that M is one-transitive, so all cone transitive CFPOs are monochromatic.

Cone transitive CFPOs are the arena for an interpretation inspired by Shelah and Truss' work. Unfortunately, there are very few Rubin complete cone transitive CFPOs. However, these methods still work when we do not have Rubin completeness, so for this chapter we drop the assumption that M is Rubin complete.

We require one additional assumption before we begin our interpretation.

Definition 2.4. *The **upwards ramification order** of x in M, written as $Ro \uparrow (x)$, is the number of upwards cones of x.*

*The **downwards ramification order** of x in M, written as $Ro \downarrow (x)$, is the number of downwards cones of x.*

Proposition 2.5. *If M is 1-transitive then for all x and y*

$$Ro \uparrow (x) = Ro \uparrow (y) \;\; \text{and} \;\; Ro \downarrow (x) = Ro \downarrow (y)$$

Proof. Any automorphism that maps x to y also maps the cones above x to the cones above y. The same is true for the cones below. □

Definition 2.6. *Let M be 1-transitive. The **upwards** (resp. **downwards**) **ramification order** of M, written as $ro \uparrow (M)$ (resp. $ro \downarrow (M)$), is equal to $Ro \uparrow (x)$ (resp. $Ro \downarrow (x)$) for some x.*

To get a sufficiently rich automorphism group we must also assume that both $ro \uparrow (M)$ and $ro \downarrow (M)$ are at least 5.

Definition 2.7. *Let K_{Cone} be the class of cone transitive CFPOs such that*

$$5 \le ro \uparrow (M) \le ro \downarrow (M)$$

for all $M \in K_{Cone}$.

3 Reconstructing Betweenness in K_{Cone}

All the CFPOs we are handling are from K_{Cone}, so are path complete, cone transitive, both $ro \uparrow (M)$ and $ro \downarrow (M)$ are greater than 4, and $ro \uparrow (M) \le ro \downarrow (M)$.

We are now ready to give the interpretation of M inside $\mathrm{Aut}(M)$. The interpretation uses pairs of subgroups isomorphic to A_5, the alternating group on five elements, to represent the points of the CFPO. A_5 is chosen because it is the smallest non-abelian finite simple group.

Definition 3.1. *Let $\bar{f}$, $\bar{f}_0$, $\bar{f}_1$, $\bar{g}$, $\bar{g}_0$ and $\bar{g}_1$ be 60-tuples from $\mathrm{Aut}(M)$.*

1. *For all $\phi \in \mathrm{Aut}(M)$, if ϕ preserves X set-wise then $\phi|^X$, the **restriction** of ϕ to X, is the map obtained by taking the union of the standard restriction, which is a partial automorphism, and the restriction of the identity to $M \setminus X$. Symbolically $\phi|^X := \phi|_X \cup id|_{M \setminus X}$. This is only a total automorphism in certain circumstances which crop up often in this chapter.*

2. *$\bar{f}(x) := \{y \in M \; : \; \exists f \in \bar{f} \; f(x) = y\}$*

3. *$A_5(\bar{f})$ is the formula that states "$\bar{f}$ satisfies the elementary diagram of A_5". This is the conjunction of formulas of the form $f_i f_j = f_k$ and $f_i f_j \ne f_k$.*

4. *$\mathrm{Comm}(\bar{f}, \bar{g})$ is the formula*

$$\mathrm{Alt}_5(\bar{f}) \wedge \mathrm{Alt}_5(\bar{g}) \wedge \bigwedge_{\substack{f_i \in \bar{f} \\ g_j \in \bar{g}}} (f_i g_j = g_j f_i)$$

5. *if $\bar{f}$ and $\bar{g}$ satisfy Alt_5 and $\phi \in \mathrm{Aut}(M)$ is any automorphism then*

$$
\begin{array}{llcl}
\bar{f} * \bar{g} & := & (f_i g_i) \qquad & \bar{f}^{\phi} & := & (\phi f_i \phi^{-1}) \\
\phi * \bar{f} & := & (\phi f_i) \qquad & \bar{f} * \phi & := & (f_i \phi)
\end{array}
$$

6. *$\mathrm{Indec}(\bar{f})$ is the formula*

$$\neg \exists \bar{g}, \bar{h}(\bar{g} * \bar{h} = \bar{f} \wedge \mathrm{Comm}(\bar{g}, \bar{h}))$$

7. *$\mathrm{Disj}(\bar{f}, \bar{g})$ is the formula*

$$\mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \mathrm{Comm}(\bar{f}, \bar{g})$$

2

8. $[\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$ *is the formula*

$$\begin{aligned}
\mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \neg\mathrm{disj}(\bar{f}, \bar{g}) \quad &\wedge \\
\neg\exists\phi[\neg\mathrm{disj}(\bar{f}^{\phi}, \bar{f}) \wedge \mathrm{disj}(\bar{g}^{\phi}, \bar{g})] \quad &\wedge \\
\neg\exists\phi(\bar{f}^{\phi} = \bar{f} \wedge \bar{g}^{\phi} \neq \bar{g}) \quad &\wedge
\end{aligned}$$

9. $[\mathrm{supp}(\bar{g}) \sqsubset \mathrm{supp}(\bar{f})]$ *is the formula*

$$[\mathrm{supp}(\bar{g}) \sqsubseteq \mathrm{supp}(\bar{f})] \wedge \neg[\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$$

10. $\mathrm{SamePD}(\bar{f}, \bar{g})$ *(Same Point and Direction) is the formula*

$$\forall\bar{h}([\mathrm{supp}(\bar{h}) \sqsubset \mathrm{supp}(\bar{f})] \leftrightarrow [\mathrm{supp}(\bar{h}) \sqsubset \mathrm{supp}(\bar{g})])$$

11. $\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$ *is the formula*

$$\mathrm{disj}(\bar{f}_0, \bar{f}_1) \wedge \forall\bar{g}\exists\bar{h}\left(\neg\mathrm{disj}(\bar{g}, \bar{h}) \wedge \left(\begin{array}{c} \mathrm{SamePD}(\bar{f}_0, \bar{h}) \\ \mathrm{SamePD}(\bar{f}_1, \bar{h}) \end{array} \vee \right)\right)$$

12. $\mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$ *is the formula*

$$\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{RepPoint}(\bar{g}_0, \bar{g}_1) \wedge$$
$$(\mathrm{SamePD}(\bar{f}_0, \bar{g}_0) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_1)) \vee (\mathrm{SamePD}(\bar{f}_0, \bar{g}_1) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_0))$$

3.1 The Domain of the Interpretation

Lemma 3.2. *If* $\mathrm{Aut}(M) \models \mathrm{Alt}_5(\bar{g})$ *holds then* $\bar{g}$ *fixes at least one point.*

Proof. Every transitive action of Alt_5 is isomorphic to its action on a coset space $[\mathrm{Alt}_5 : H]$ for some $H \leq A_5$. The subgroups of A_5 can have orders 1, 2, 3, 4, 5, 6, 10, 12 and 60 and hence the possible values for $|\bar{g}(x)|$ are 60, 30, 20, 15, 12, 10, 6, 5 and 1.

Every element of $\bar{g}$ has finite order so for all x we know that $\bar{g}(x)$ is an antichain. Pick one x such that $|\bar{g}(x)| \neq 1$ (possible, as A_5 is not the identity), so there are g_i that act non-trivially.

Let

$$S := \bigcup_{x_i, x_j \in \bar{g}(x)} \mathrm{Path}\langle x_i, x_j \rangle^{-}$$

Since each $\mathrm{Path}\langle x_i, x_j \rangle^{-}$ is finite and $\bar{g}(x)$ is finite, S is also finite, and therefore must be a CFPO_n for some n. In the previous paper we showed that there was a tree T such that $\mathrm{Aut}(S) \cong_P \mathrm{Aut}(T)$. The root of T is fixed by every automorphism of S, and hence by every element of $\bar{g}$. $\square$

Lemma 3.3. *If* $\mathrm{supp}(\bar{f})$ *and* $\mathrm{supp}(\bar{g})$ *are disjoint then* $\mathrm{Aut}(M) \models \mathrm{Comm}(\bar{f}, \bar{g})$.

Definition 3.4. *Let* $\bar{f}$ *be such that* $\mathrm{Aut}(M) \models A_5(\bar{f})$ *and* C_i *are the connected components of* $\mathrm{supp}(\bar{f})$. *We say that* $E \subseteq M$ *is an extended connected component of* $\mathrm{supp}(\bar{f})$ *if:*

1. E *contains a union of* C_i *and at most one element from* $M \setminus \mathrm{supp}(\bar{f})$, *which we call* e;

2. *if* $C \subseteq E$ *then* $\bar{f}(C) \subseteq E$;

3. *if* e *exists then* E *contains at least two connected components,* C_0 *and* C_1, *and* $\{e\} = \mathrm{Path}\langle C_0, C_1 \rangle$; *and*

4. *if* D *satisfies conditions 1-3 and* $E \cap D \neq \emptyset$ *then* $E \subseteq D$.

Lemma 3.5. *If* X *is an extended connected component of some* $\mathrm{supp}(\bar{f})$ *then* $\mathrm{Path}\langle X, M \setminus X \rangle$ *is a singleton.*

Proof. Condition 2 of Definition 3.4 shows that X is preserved setwise by $\bar{f}$, so by Lemma 3.10 of Part 1. There are x and y such that $\mathrm{Path}\langle X, M \setminus X \rangle = \mathrm{Path}\langle x, y \rangle$. Both x and y are fixed by $\bar{f}$, so $x, y \in M \setminus X$.

Suppose one of the cones above x intersects X and one of the cones below x intersects X. Let U be the upwards extended cone and let D be the downwards extended cone. $\bar{f}(U) \cap \bar{f}(D) = \emptyset$, as $\bar{f}$ fixes x, so $\bar{f}(U) \cap X$ satisfies Conditions 1-3, and does not contain X giving a contradiction.

Therefore we may assume that X is contained in extended cones above x. Let y_0 and y_1 lie in different extended cones below x. The definition of extended cone guarantees that $\mathrm{Path}\langle x, y_0 \rangle \cap \mathrm{Path}\langle x, y_1 \rangle = \{x\}$, so $\mathrm{Path}\langle x, y \rangle = \{x\}$. $\square$

Lemma 3.6. *Let $\bar{f}$ satisfy* Alt_5. *If we partition* $\mathrm{supp}(\bar{f})$ *into two collections of extended connected components, which we will call X and Y, then $(f_i|^X)$ and $(f_i|^Y)$ satisfy* Alt_5.

Proof. First of all, we must show that this lemma makes sense, i.e. $\bar{f}$ preserves the extended connected components of $\mathrm{supp}(\bar{f})$ set-wise and therefore $f_i|^X$ and $f_i|^Y$ are automorphisms.

Since the supports of $(f_i|^X)$ and $(f_i|^Y)$ are disjoint, $\mathrm{Comm}((f_i|^X), (f_i|^Y))$ holds. We consider the positive statements of the formula A_5 that $\bar{f}$ satisfies, which are of the form $f_i f_j = f_k$.

Since $f_i = f_i|^X f_i|^Y$ for all i we can deduce that

$$f_i|^X f_j|^X f_i|^Y f_j|^Y = f_k|^X f_k|^Y$$

and since $(f_\alpha|^X f_\alpha|^Y)|^X = f_\alpha|^X$ we conclude that $(f_i|^X)$ and $(f_i|^Y)$ satisfy all the positive statements of Alt_5. We now consider the negative statements, those of the form $f_i f_j \neq f_k$.

Repeating the argument for the positive statements allows us to deduce

$$f_i|^X f_j|^X f_i|^Y f_j|^Y \neq f_k|^X f_k|^Y$$

which only guarantees that at least one of $f_i|^X f_j|^X \neq f_k|^X$ or $f_i|^Y f_j|^Y \neq f_k|^Y$. Without loss of generality we assume that $f_i|^Y f_j|^Y \neq f_k|^Y$. In A_5 there is the positive statement $f_i f_j = f_l$ for some $f_l \neq f_k$, so if $f_i|^X f_j|^X = f_k|^X$, then $f_k|^X = f_l|^X$.

We define the homomorphism

$$\Phi : \begin{cases} \bar{f} & \to & (f_i|^X) \\ f_i & \mapsto & f_i|^X \end{cases}$$

$\Phi^{-1}(id)$ is a normal subgroup of $\bar{f}$. We have just found distinct f_k and f_l such that $f_k|^X = f_l|^X$, so since A_5 is simple, this means that $f_i|^X = id$ for all $f_i \in \bar{f}$, contradicting the fact that $X \cap \mathrm{supp}(\bar{f}) \neq \emptyset$.

Therefore if $A_5(\bar{f})$ then $A_5((f_i|^X))$ and $A_5((f_i|^Y))$. $\square$

Lemma 3.7. *If $\bar{g} * \bar{h} = \bar{f}$ and $\mathrm{Comm}(\bar{g}, \bar{h})$ then $\mathrm{supp}(\bar{g}), \mathrm{supp}(\bar{h}) \subseteq \mathrm{supp}(\bar{f})$.*

Proof. Suppose there is an x such that $g_j(x) \neq x$ for some j and $f_i(x) = x$ for all i. Therefore

$$\forall i \qquad h_i g_i(x) \;=\; x$$

$$\forall i \qquad g_i(x) \;=\; h_i^{-1}(x)$$

There are g_j and g_k such that $g_j g_k(x) \neq g_k g_j(x)$ as A_5 is non-abelian, and if we substitute h_j^{-1} for g_j we find that

$$h_j^{-1} g_k(x) \neq g_k h_j^{-1}(x)$$

contradicting $M \models \mathrm{Comm}(\bar{g}, \bar{h})$. $\square$

Lemma 3.8. *Let X and Y be extended connected components of $\mathrm{supp}(\bar{f})$ and $\mathrm{supp}(\bar{g})$. If $\mathrm{Comm}(\bar{f}, \bar{g})$ and $|X \cap Y| \geq 1$ then either $X \subseteq Y$ or $Y \subseteq X$.*

Proof. Let $\{x\} = \mathrm{Path}\langle X, M \setminus X \rangle$ and $\{y\} = \mathrm{Path}\langle Y, M \setminus Y \rangle$. These are singletons by Lemma 3.5. Suppose $X \nsubseteq Y$ and $Y \nsubseteq X$.

First suppose that $x = y$. This means that $\mathrm{Path}\langle X, Y \rangle = \{x\}$, and that X and Y are entirely contained in the upwards and downwards extended cones of x, as illustrated in Figure 4.1.
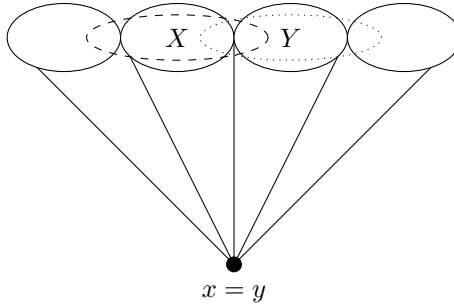

Figure 1: If $x = y$ in Lemma 3.8

Recall Definition 3.4, and note that $\bar{g}(X \cap Y)$ satisfies conditions 1 and 3 because both X and Y do, and by definition it satisfies condition 2. Therefore $Y \subseteq \bar{g}(X \cap Y)$. Thus if $\bar{g}(X \cap Y) \subseteq X$ then $Y \subseteq X$ and we are done. Similarly if $\bar{f}(X \cap Y) \subseteq Y$ then $X \subseteq Y$ and we are done.

We now suppose that there is a $z \in X \cap Y$ such that $\bar{f}(z) \nsubseteq Y$ and $\bar{g}(z) \nsubseteq X$. Let C_z be the extended cone of x that contains z. We consider the action of $\bar{f}$ and $\bar{g}$ on the set $\bar{f}(C_z) \cup \bar{g}(C_z)$.

Let $f_i \in \bar{f}$ map C_z into $X \setminus Y$ and let g_j map C_z into $Y \setminus X$. Then

$$f_i g_j(C_z) = g_j(C_z) \text{ and } g_j f_i(C_z) = f_i(C_z)$$

contradicting the assumption that $\mathrm{Aut}(M) \models \mathrm{Comm}(\bar{f}, \bar{g})$. This is depicted in Figure 20.
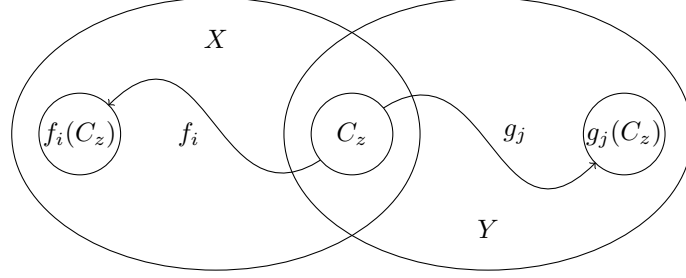

Figure 2: Images of C_z

Now suppose that $x \neq y$. Suppose $x \notin Y$ and $y \in X$, and let $z \in Y$. By definition, $y \in \mathrm{Path}\langle z, x \rangle$, and since x is an endpoint of that path, $\mathrm{Path}\langle z, x \rangle \subseteq X$, and so $z \in X$. This is depicted in Figure 21.
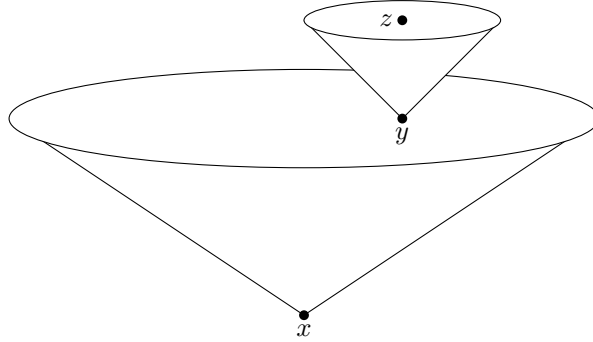

Figure 3: $x \notin Y$ and $y \in X$

If both $x \notin Y$ and $y \notin X$ then $\mathrm{Path}\langle x, y \rangle \subseteq M \setminus (X \cup Y)$. This is depicted in Figure 22. Let $z \in Y$. By definition $y \in \mathrm{Path}\langle x, z \rangle$ and since $\mathrm{Path}\langle x, y \rangle \nsubseteq X$, we know that $z \notin X$. Similarly, if $z \in X$ then $z \notin Y$, contradicting the assumption that $X \cap Y \neq \emptyset$.
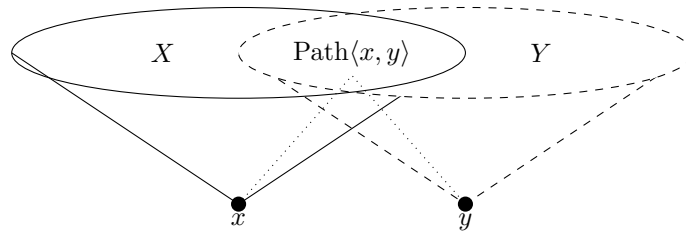

Figure 4: $x \notin Y$ and $y \notin X$

We therefore suppose that $x \in Y$ and $y \in X$.

$x \in \text{Path}\langle y, f_i(y)\rangle$ for any f_i, as otherwise X will not be an extended connected component.

Path-betweenness is preserved by automorphisms, so $g_j(x) \in \text{Path}\langle g_j(y), g_j f_i(y)\rangle$ and $\bar{f}$ and $\bar{g}$ commute, and y is fixed by $\bar{g}$, hence $g_j(x) \in \text{Path}\langle y, f_i(y)\rangle$. By symmetry $y \in \text{Path}\langle x, g_j(x)\rangle$ and $f_i(y) \in \text{Path}\langle x, g_j(x)\rangle$. From these facts we can deduce the path-configuration of x, y, $g_j(x)$ and $f_i(y)$.

$$\overset{\displaystyle \text{Path}\langle x, y\rangle}{\underset{\displaystyle x \qquad\qquad\qquad\qquad y}{\rule{6cm}{0.4pt}}}$$
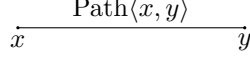
Figure 5: $\text{Path}\langle x, y\rangle$

Since $y \in \text{Path}\langle x, g_j(x)\rangle$ and $x \in \text{Path}\langle y, f_i(y)\rangle$ we may add to Figure 23 $f_i(y)$ and $g_j(x)$ to obtain Figure 24.

$$\underset{\displaystyle f_i(y) \qquad\qquad\qquad x \qquad\qquad\qquad y \qquad\qquad\qquad g_j(x)}{\rule{11cm}{0.4pt}}$$

Figure 6: $\text{Path}\langle x, y\rangle$, $f_i(y)$ and $g_j(x)$

But we also know that $f_i(y) \in \text{Path}\langle x, g_j(x)\rangle$, so we deduce that $f_i(y) = x$. Similarly $g_j(x) \in \text{Path}\langle y, f_i(y)\rangle$ shows that $g_j(x) = y$. This contradicts the fact that $\bar{f}$ fixes x and $\bar{g}$ fixes y, so we conclude that either $X \subseteq Y$ or $Y \subseteq X$. $\qquad\square$

Lemma 3.9. *If* $\text{Aut}(M) \models \text{Comm}(\bar{f}, \bar{g})$ *and* $\text{supp}(\bar{f}) \cap \text{supp}(\bar{g}) \neq \emptyset$ *then* $\bar{f} * \bar{g}$ *has an orbit of length 20 in* $\text{supp}(\bar{f}) \cap \text{supp}(\bar{g})$. *If* $\bar{f} * \bar{g}$ *has an orbit of length 20 then it also has a non-trivial orbit of some length other than 20.*

Proof. Lemma 3.5 of [4] is:

"Suppose that $\bar{f}, \bar{g}$ are subgroups of $\text{Sym}(\mathcal{X})$ isomorphic to A_5 (in the specified listings) which centralize each other, and such that $\langle \bar{f}, \bar{g}\rangle$ is transitive on $\mathcal{X}$. Then $\bar{f} * \bar{g}$ has an orbit of length 20. Moreover, if $\bar{f} * \bar{g}$ has an orbit of length 20 then is also has an orbit of some other length greater than 1."

Let $\{A_i : i \in I\}$ be the ECC of $\text{supp}(\bar{f})$ and let $\{B_j : j \in J\}$ be the ECC of $\text{supp}(\bar{g})$. Lemma 3.8 shows that if $A_i \cap B_j \neq \emptyset$ then $A_i \subseteq B_j$ or $B_j \subseteq A_i$.

Pick one such A and B, and without loss of generality assume that $A \subseteq B$. Let X be a connected component of A.
$$\mathcal{X} := \langle \bar{f}, \bar{g}\rangle(X)$$
Each member of $\mathcal{X}$ is a translate of X.

We define $\phi_f : \bar{f} \to \text{Sym}(\mathcal{X})$ as follows: $\phi_f(f_i) = (X \mapsto f_i(X))$. This is a homomorphism, and since A_5 is simple, so ϕ's kernel is trivial, and $\phi_f(\bar{f}) \cong A_5$. Similarly, if we define $\phi_g : \bar{g} \to \text{Sym}(\mathcal{X})$ as follows: $\phi_g(g_i) = (X \mapsto g_i(X))$. then $\phi_g(\bar{g}) \cong A_5$.

The 'specified listings' in Shelah and Truss' Lemma 3.5 refers to the fact that the formula $A_5(\bar{f})$ will be different depending on how we enumerate A_5. For example, we could insist that f_0 is the identity, and this would give a different formula to if we insisted that f_5 is the identity. Our formula A_5 is fixed so we need not worry about this assumption.

$\langle \phi_f(\bar{f}), \phi_g(\bar{g})\rangle$ is transitive on $\mathcal{X}$ since $\mathcal{X}$ is an orbit of $\langle \bar{f}, \bar{g}\rangle$.

Therefore Lemma 3.5 of [4] is applicable to $\mathcal{X}$. $\qquad\square$

Lemma 3.10. *If* $\text{Aut}(M) \models A_5(\bar{f})$ *then no orbit of* $\bar{f}$ *has length 60.*

Proof. Let $x \in M$ be such that $|\bar{f}(x)| = 60$, and let $\{x_0, \dots x_{59}\}$ be an enumeration of $\bar{f}(x)$. Take $X(5, 60, \mathbb{Z})$, and pick an arbitrary $z \in X(5, 60, \mathbb{Z})$, and label the successors of z as $z_0, \dots z_{59}$. For each $f_i \in \bar{f}$, let $g_i \in \text{Aut}(X(5, 60, \mathbb{Z}))$ be induced by the partial automorphism

$$z_n \mapsto z_m \text{ if } f_i(x_n) = x_m$$

$\text{Aut}(X(5, 60, \mathbb{Z})) \models A_5(\bar{g})$ and $\bar{g}$ has an orbit of length 60. Let C_i be the extended cone of z that contains z_i.

6

For each $y \in \bar{g}(z)$ there is a unique $g_i \in \bar{g}$ such that $g_i(z) = y$, so we may label $\bar{g}(x)$ by elements of $\bar{g}$. In this way, we can view the action of $\bar{g}$ on $\bar{g}(x)$ as left multiplication.

We define $\bar{h}$ on each $g \in \bar{h}$ as follows:

$$h_i : g \mapsto gg_i^{-1}$$

This $\bar{h}$ commutes with $\bar{g}$, as

$$
\begin{aligned}
h_i g_j(g) &= h_i(g_j g) \\
&= (g_j g)g_i^{-1} \\
&= g_j(gg_i^{-1}) \\
&= g_j(h_i(g)) \\
&= g_j h_i(g)
\end{aligned}
$$

We may extend each $h_i \in \bar{h}$ to an automorphism of $X(5, 60, \mathbb{Z})$ as follows

$$
y \mapsto \begin{cases} y & z \in X(5, 60, \mathbb{Z}) \setminus \bigcup_{i<60} C_i \\ g_j(y) & y \in C_k \text{ and } g_j : C_k \mapsto h_i(C_k) \end{cases}
$$

We now have a $\bar{h} \in \mathrm{Aut}(X(5, 60, \mathbb{Z}))$ such that $\mathrm{Aut}(X(5, 60, \mathbb{Z})) \models \mathrm{Comm}(\bar{g}, \bar{h})$

Remember that $\mathrm{id} \in \bar{g}$ and consider $\bar{g} * \bar{h}$. For all $g_i h_i \in \bar{g} * \bar{h}$

$$
\begin{aligned}
g_i h_i(\mathrm{id}) &= g_i \mathrm{id} g_i^{-1} \\
&= \mathrm{id}
\end{aligned}
$$

Since $\mathrm{id}(x)$ was labelled as id, this means that $x \in \mathrm{supp}(\bar{g}) \cap \mathrm{supp}(\bar{h})$, but $x \notin \mathrm{supp}(\bar{g} * \bar{h})$, contradicting Lemma 3.7. $\qquad \square$

Lemma 3.11. *If* $\mathrm{Aut}(M) \models A_5(\bar{f})$ *and there is an* $x \in M$ *such that* $|\bar{f}(x)| = 30$ *then there are* $\bar{g}$ *and* $\bar{h}$ *such that* $\mathrm{Aut}(M) \models \mathrm{Comm}(\bar{g}, \bar{h})$ *and* $\bar{f} = \bar{g} * \bar{h}$.

Proof. Let $\mathrm{Aut}(M) \models A_5(\bar{f})$ be such that there is an $x \in M$ such that $|\bar{f}(x)| = 30$. Let X be the ECC of $\mathrm{supp}(\bar{f})$ that contains x.

Let G and H be subgroups of A_5 such that $|G| = 12$ and $|H| = 10$. There is a transitive action of A_5 on $\{aG : a \in A_5\} \times \{bH : b \in A_5\}$ which is isomorphic (as permutation groups) to $\bar{f}$'s action on $\bar{f}(x)$.

We may therefore label each cone of X as (aG, bH). We define $\bar{g}, \bar{h} \in \mathrm{Aut}(M)$ as follows:

$$
g_i : z \mapsto \begin{cases} f_i(z) & z \in M \setminus X \\ f_j(z) & f_j((aG, bH)) = (g_i aG, bH) \end{cases}
$$

$$
h_i : z \mapsto \begin{cases} z & z \in M \setminus X \\ f_j(z) & f_j((aG, bH)) = (aG, h_i bH) \end{cases}
$$

If $\bar{g}$ and $\bar{h}$ are not well-defined then there is an f_i such that $f_i((aG, bH)) = (aG, bH)$ but there is a $z \in (aG, bH)$ such that $f_i(z) \neq z$. However $\bar{f}$ acts transitively on the (aG, bH), so $|\bar{f}(z)| = 60$. Lemma 3.10 shows that no such $\bar{f}$ exists.

If $z \in M \setminus X$ then $g_i h_i(z) = g_i(z) = f_i(z)$, so $(\bar{g} * \bar{h})|_{M \setminus X} = \bar{f}|_{M \setminus X}$. If $z \in (aG, bH)$ then $g_i h_i((aG, bH)) = (g_i(a)G, h_i(b)H) = f_i((aG, bH))$, therefore $g_i h_i(z) = f_i(z)$, and so $(\bar{g} * \bar{h})|_X = \bar{f}|_X$.

Together, we now have $(\bar{g} * \bar{h}) = \bar{f}$, so the lemma is proved. $\qquad \square$

Proposition 3.12. $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f})$ *if and only if* $\mathrm{supp}(\bar{f})$ *has exactly one extended connected component and every orbit has less than 30 members.*

Proof. First we prove that if $\mathrm{supp}(\bar{f})$ has exactly one extended connected component and every orbit has less than 30 members then $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f})$ by contradiction. Let $\bar{g}$ and $\bar{h}$ witness the fact that $\bar{f}$ does not satisfy Indec, i.e. $\bar{f} = \bar{g} * \bar{h}$ and $\mathrm{Aut}(M) \models \mathrm{Comm}(\bar{g}, \bar{h})$.

If $\mathrm{supp}(\bar{g}) \cap \mathrm{supp}(\bar{h}) = \emptyset$ then $\bar{f}$ fixes $\mathrm{supp}(\bar{g})$ and $\mathrm{supp}(\bar{h})$ setwise, and hence $\mathrm{supp}(\bar{g})$ and $\mathrm{supp}(\bar{h})$ lie in different ECCs of $\mathrm{supp}(\bar{f})$.

If $\mathrm{supp}(\bar{g}) \cap \mathrm{supp}(\bar{h}) \neq \emptyset$ then Lemma 3.9 shows that $\bar{g} * \bar{h}$ has an orbit of length at least 20. If $\bar{g} * \bar{h}$ has an orbit of length 20 then there is also another orbit of length other than 20. Since the length is other than 20, this other orbit cannot lie in the same ECC as the orbit of length 20.

Therefore if $\mathrm{supp}(\bar{f})$ has exactly one extended connected component and every orbit has less than 30 members then $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f})$. We now turn our attention to the other direction, which we also do by contradiction.

Suppose $\mathrm{supp}(\bar{f})$ has multiple extended connected components. We let X be one of these extended connected components and consider $\bar{f}|^X$ and $\bar{f}|^{M \setminus X}$. These two both satisfy Alt_5 (by Lemma 3.6) and their supports are disjoint, so they satisfy Comm. Finally $\bar{f}|^X * \bar{f}|^{M \setminus X} = \bar{f}$, showing that $\bar{f}|^X$ and $\bar{f}|^{M \setminus X}$ witness the fact that $\bar{f}$ does not satisfy Indec.

Lemma 3.10 shows that $\bar{f}$ cannot have an orbit of length 60. Lemma 3.11 shows that if $\bar{f}$ has an orbit of length 30 then $\mathrm{Aut}(M) \models \neg\mathrm{Indec}(\bar{f})$. $\qquad\square$

Lemma 3.13. *If* $\mathrm{Aut}(M) \models disj(\bar{f}, \bar{g})$ *then* $\mathrm{supp}(\bar{f}) \cap \mathrm{supp}(\bar{g}) = \emptyset$.

Proof. Suppose $\mathrm{supp}(\bar{f}) \cap \mathrm{supp}(\bar{g}) \neq \emptyset$. By Lemma 3.8 either

$$\mathrm{supp}(\bar{f}) \subseteq \mathrm{supp}(\bar{g}) \text{ or } \mathrm{supp}(\bar{f}) \subseteq \mathrm{supp}(\bar{g})$$

Now assume that $\mathrm{supp}(\bar{f}) \subsetneq \mathrm{supp}(\bar{g})$ and let $z \in \mathrm{supp}(\bar{f})$.

$\mathrm{Path}\langle \mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f}) \rangle$ is a singleton, as $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f})$. Let

$$\{x_f\} := \mathrm{Path}\langle \mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f}) \rangle$$

$x_f \notin \mathrm{supp}(\bar{f})$, but since $\mathrm{supp}(\bar{f}) \subsetneq \mathrm{supp}(\bar{g})$, we know that $x_f \in \mathrm{supp}(\bar{g})$.

Let $g_i \in \bar{g}$. By definition $x_f \in \mathrm{Path}\langle g_i^{-1}(x_f), z \rangle$. Since paths are preserved by automorphisms, this translates to

$$g_i(x_f) \in \mathrm{Path}\langle x_f, g_i(z) \rangle$$

Thus if $g_i \neq \mathrm{id}$ then $g_i(z) \notin \mathrm{supp}(\bar{f})$, i.e. $f_j g_i(z) = g_i(z)$ for all j, but since $z \in \mathrm{supp}(\bar{f})$ there is a k such that $f_k(z) \neq z$. This is depicted in Figure 25.

$$
\begin{aligned}
g_i(z) &= f_k g_i(z) \\
&= g_i f_k(z) \\
&\neq g_i(z)
\end{aligned}
$$

This is a contradiction. Therefore if $\mathrm{supp}(\bar{f}) \cap \mathrm{supp}(\bar{g}) \neq \emptyset$ then $\mathrm{supp}(\bar{f}) = \mathrm{supp}(\bar{g})$.
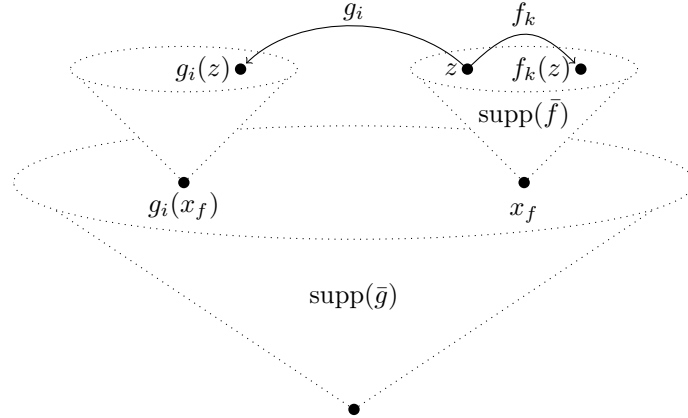

Figure 7: $\mathrm{supp}(\bar{f}) \subsetneq \mathrm{supp}(\bar{g})$

Suppose that $\mathrm{supp}(\bar{f}) = \mathrm{supp}(\bar{g})$. Again, $\mathrm{Path}\langle \mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f}) \rangle$ is a singleton, as $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f})$ so we let

$$\{x_f\} := \mathrm{Path}\langle \mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f}) \rangle$$

Both $\bar{f}$ and $\bar{g}$ must act transitively on the same antichain of immediate successors or predecessors of x_f, which $\bar{f} * \bar{g}$ must also act on. Since $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f})$ and $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{g})$, Proposition 3.12 shows that this antichain must have less than 30 members, but Lemma 3.9 showed that $\bar{f} * \bar{g}$ must have an orbit of at least 20 members.

Lemma 3.9 also showed that if $\bar{f} * \bar{g}$ has an orbit of length 20 then there was another orbit. Therefore $\bar{f}$ acts transitively on a set with strictly more than 20 elements, and hence at least 30, which contradicts Proposition 3.12.

Therefore $\mathrm{supp}(\bar{f}) \cap \mathrm{supp}(\bar{g}) = \emptyset$. $\qquad\square$

Lemma 3.14. *Recall that* $[\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$ *is the formula*

$$\begin{aligned}
\mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \neg\mathrm{disj}(\bar{f}, \bar{g}) &\quad \wedge \\
\forall\phi[\mathrm{disj}(\bar{g}^\phi, \bar{g}) \to \mathrm{disj}(\bar{f}^\phi, \bar{f})] &\quad \wedge \\
\forall\phi(\bar{g}^\phi \neq \bar{g} \to \bar{f}^\phi = \bar{f}) &\quad \wedge
\end{aligned}$$

If $\bar{f}$ *and* $\bar{g}$ *satisfy this formula then the support of* $\bar{g}$ *is contained in the support of* $\bar{f}$.

Proof. The two sentences

$$\forall\bar{f},\bar{g}\left(\begin{array}{c} (\forall\phi[\mathrm{disj}(\bar{g}^\phi, \bar{g}) \to \mathrm{disj}(\bar{f}^\phi, \bar{f})]) \\ \leftrightarrow \\ (\neg\exists\phi[\neg\mathrm{disj}(\bar{f}^\phi, \bar{f}) \wedge \mathrm{disj}(\bar{g}^\phi, \bar{g})]) \end{array}\right) \text{ and } \forall\bar{f},\bar{g}\left(\begin{array}{c} (\forall\phi(\bar{g}^\phi \neq \bar{g} \to \bar{f}^\phi = \bar{f})) \\ \leftrightarrow \\ (\neg\exists\phi(\bar{f}^\phi = \bar{f} \wedge \bar{g}^\phi \neq \bar{g})) \end{array}\right)$$

are tautologies, so the formula given here is equivalent to the one given in Definition 3.1.

Suppose that $\bar{f}$ and $\bar{g}$ are such that

$$\mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g}) \wedge \neg\mathrm{disj}(\bar{f}, \bar{g})$$

This means that $\mathrm{supp}(\bar{f})$ and $\mathrm{supp}(\bar{g})$ each have exactly one ECC, which have a non-empty intersection. We define

$$\begin{aligned}
x_f &:= \mathrm{Path}\langle\mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f})\rangle \\
x_g &:= \mathrm{Path}\langle\mathrm{supp}(\bar{g}), M \setminus \mathrm{supp}(\bar{g})\rangle
\end{aligned}$$

If $\mathrm{supp}(\bar{f}) = \mathrm{supp}(\bar{g})$ then $\mathrm{supp}(\bar{f}^\phi) = \mathrm{supp}(\bar{g}^\phi)$ for all $\phi \in \mathrm{Aut}(M)$. Therefore for all $\phi \in \mathrm{Aut}(M)$

$$\mathrm{Aut}(M) \models (\mathrm{disj}(\bar{f}^\phi, \bar{f}) \leftrightarrow \mathrm{disj}(\bar{g}^\phi, \bar{g}))$$

and

$$\mathrm{Aut}(M) \models (\bar{g}^\phi \neq \bar{g} \leftrightarrow \bar{f}^\phi = \bar{f})$$

Thus $\mathrm{Aut}(M) \models [\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$.

We now suppose that $\mathrm{supp}(\bar{f}) \neq \mathrm{supp}(\bar{g})$. In Case 1 we consider $\mathrm{supp}(\bar{g}) \subsetneq \mathrm{supp}(\bar{f})$. In Case 3 we consider $\mathrm{supp}(\bar{f}) \subsetneq \mathrm{supp}(\bar{g})$. If neither $\mathrm{supp}(\bar{g}) \subsetneq \mathrm{supp}(\bar{f})$ nor $\mathrm{supp}(\bar{f}) \subsetneq \mathrm{supp}(\bar{g})$ then we are either in Case 2, where $x_f \neq x_g$, or Case 4 where $x_f = x_g$.
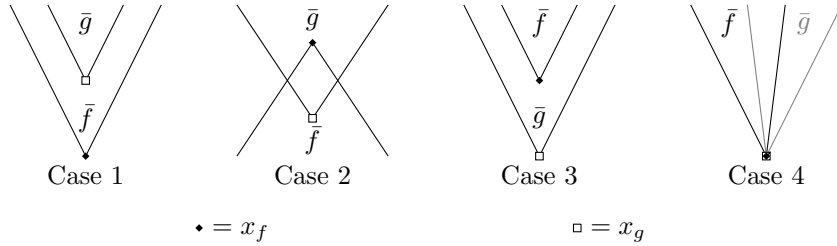

$\bullet = x_f$ $\qquad\qquad\qquad$ $\square = x_g$

Figure 8: Cases of Lemma 3.14

In Case 3 we must prove that $\mathrm{Aut}(M) \models [\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$, while in Cases 1 and 2, we must show that the converse holds. Finally, in Case 4 we show that $\mathrm{Aut}(M) \models [\mathrm{supp}(\bar{f}) \subseteq \mathrm{supp}(\bar{g})]$ if and only if $\mathrm{supp}(\bar{f}) = \mathrm{supp}(\bar{g})$.

Case 1 Since x_f is moved by $\bar{g}$ there is an x_f' such that x_f and x_f' lie in the same $\bar{g}$-orbit and $x_f \neq x_f'$. Let ϕ be an automorphism that switches x_f and x_f', but fixes anything that it does not have to move. If $z \in \mathrm{supp}(\bar{f})$ then $\phi(z) \notin \mathrm{supp}(\bar{f})$ and so $\mathrm{disj}(\bar{f}^\phi, \bar{f})$. Since $\mathrm{Path}\langle x_f, x_f'\rangle \subseteq \mathrm{supp}(\bar{g})$ we know that $\mathrm{supp}(\bar{g}) = \mathrm{supp}(\bar{g}^\phi)$ and therefore $\neg\mathrm{disj}(\bar{g}^\phi, \bar{g})$

Thus ϕ witnesses the fact that $\bar{f}$ and $\bar{g}$ do not satisfy $[\mathrm{supp}(\bar{g}) \sqsubseteq \mathrm{supp}(\bar{f})]$.

Case 2 Let x_f' be such that $x_f \in \mathrm{Path}\langle x_g, x_f'\rangle$ and $x_f \parallel x_f'$. Since $X(n, m, \mathbb{Z})$ is 1-transitive there is an automorphism ϕ such that $\phi(x_f) = x_f'$. We know that $\mathrm{disj}(\bar{f}^\phi, \bar{f})$ as

$$\mathrm{Path}\langle\mathrm{supp}(\phi * \bar{f}), \mathrm{supp}(\bar{f})\rangle = \mathrm{Path}\langle f, f'\rangle$$

which cannot be empty, as $x_f \parallel x_f'$. Since $x_f \in \mathrm{Path}\langle x_g, x_f'\rangle$ and $x_f \in \mathrm{supp}(\bar{g})$ the support of $\bar{g}$ must contain x_f'. However x_f' is clearly contained in $\mathrm{supp}(\phi * \bar{g})$, so $\neg\mathrm{disj}(\phi * \bar{g}, \bar{g})$.

Thus ϕ witnesses the fact that $\bar{f}$ and $\bar{g}$ do not satisfy $[\mathrm{supp}(\bar{g}) \sqsubseteq \mathrm{supp}(\bar{f})]$.

9

Case 3 For a contradiction, assume that

$$\mathrm{Aut}(M) \models \exists\phi[\mathrm{disj}(\bar{f}^\phi, \bar{f}) \wedge \neg\mathrm{disj}(\bar{g}^\phi, \bar{g})]$$

and let ϕ witness this. Since $\mathrm{disj}(\bar{f}^\phi, \bar{f})$ holds, and $\mathrm{supp}(\bar{g})$ is contained in $\mathrm{supp}(\bar{f})$, we know that $\mathrm{disj}(\bar{g}^\phi, \bar{g})$, giving a contradiction.

Now assume that

$$\mathrm{Aut}(M) \models \exists\phi(\bar{f}^\phi = \bar{f} \wedge \bar{g}^\phi \neq \bar{g})$$

Let C_0, C_1 be two of the cones of x_f that are contained in the support of $\bar{f}$ and let $f_i \in \bar{f}$ map C_0 to C_1. Since $\bar{g}^\phi \neq \bar{g}$, there is an $x \in \mathrm{supp}(\bar{g})$ such that $\phi(x) \neq x$. We suppose without loss of generality that $x \in C_0$.

If $\phi(x) \notin C_1$ then f_i^ϕ will map x to $f_i\phi(x) \neq f_i(x)$ and so $\bar{f}^\phi \neq \bar{f}$. If $\phi(x) \in C_1$ then conjugation by ϕ will at least switch the roles C_0 and C_1, and so $\bar{f}^\phi \neq \bar{f}$.

Case 4 In this case, $x_f = x_g$. Let $C_0^f, \ldots$ be the cones of f that are contained in $\bar{f}$, and let $C_0^g, \ldots$ be the cones of x_g that are contained in $\bar{g}$. We may pick our indices such that $C_i^f \in \mathrm{supp}(\bar{f}) \cap \mathrm{supp}(\bar{g})$ if and only if $C_i^g \in \mathrm{supp}(\bar{f}) \cap \mathrm{supp}(\bar{g})$.

Assume that only one C_i^f is not in the intersection of the supports, and assume without loss of generality that this is C_0^f. Let $\phi \in \mathrm{Aut}(M)$ be such that $\mathrm{supp}(\phi) \subsetneq C_0^g$ and. Then

$$\mathrm{Aut}(M) \models (\bar{f}^\phi = \bar{f} \wedge \bar{g}^\phi \neq \bar{g})$$

showing that $\bar{f}$ and $\bar{g}$ do not satisfy $[\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$.

Now we assume that more that one C_i^f is not in the intersection of the supports, without loss of generality C_0^f and C_1^f. Let $\phi \in \mathrm{Aut}(M)$ be such that ϕ swaps C_0^g and C_1^g and fixes everything else point-wise. Since ϕ fixes $\mathrm{supp}(\bar{f})$ point-wise, $\mathrm{Aut}(M) \models \bar{f}^\phi = \bar{f}$.

Now consider a elements of $\bar{g}$ which switches C_0^g and C_2^G. The corresponding elements of $\bar{g}^\phi$ will switch C_1^g and C_2^g, and so $\mathrm{Aut}(M) \models \bar{g}^\phi \neq \bar{g}$. $\qquad\square$

Corollary 3.15. *Recall that* $[\mathrm{supp}(\bar{g}) \sqsubset \mathrm{supp}(\bar{f})]$ *is the formula*

$$[\mathrm{supp}(\bar{g}) \sqsubseteq \mathrm{supp}(\bar{f})] \wedge \neg[\mathrm{supp}(\bar{f}) \sqsubseteq \mathrm{supp}(\bar{g})]$$

$\mathrm{Aut}(M) \models [\mathrm{supp}(\bar{g}) \sqsubset \mathrm{supp}(\bar{f})]$ *if and only if* $\mathrm{supp}(\bar{g})$ *is properly contained in* $\mathrm{supp}(\bar{f})$.

Definition 3.16. *Let* $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{f}) \wedge \mathrm{Indec}(\bar{g})$ *and let*

$$x_f := \mathrm{Path}\langle\mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f})\rangle$$
$$x_g := \mathrm{Path}\langle\mathrm{supp}(\bar{g}), M \setminus \mathrm{supp}(\bar{g})\rangle$$

We say that $\bar{f}$ *and* $\bar{g}$ *have the **same direction**, or act in the same direction if*

$$\exists y \in \mathrm{supp}(\bar{f})\,(x_f < y) \Leftrightarrow \exists z \in \mathrm{supp}(\bar{g})\,(x_g < z)$$

We say that $\bar{f}$ *and* $\bar{g}$ *have **different directions**, or act in different directions if*

$$\exists y \in \mathrm{supp}(\bar{f})\,x_f < y \Leftrightarrow \exists z \in \mathrm{supp}(\bar{g})\,(x_g > z)$$

Lemma 3.17. *Recall that* $\mathrm{SamePD}(\bar{f}, \bar{g})$ *is the formula*

$$\forall\bar{h}([\mathrm{supp}(\bar{h}) \sqsubset \mathrm{supp}(\bar{f})] \leftrightarrow [\mathrm{supp}(\bar{h}) \sqsubset \mathrm{supp}(\bar{g})])$$

Let

$$\{x_f\} := \mathrm{Path}\langle\mathrm{supp}(\bar{f}), M \setminus \mathrm{supp}(\bar{f})\rangle$$
$$\{x_g\} := \mathrm{Path}\langle\mathrm{supp}(\bar{g}), M \setminus \mathrm{supp}(\bar{g})\rangle$$

If

$$\mathrm{Aut}(M) \models \mathrm{SamePD}(\bar{f}, \bar{g})$$

then $f = g$ *and* $\bar{f}$ *and* $\bar{g}$ *have the same direction.*

Proof. Suppose $\mathrm{Aut}(M) \models \mathrm{SamePD}(\bar{f}, \bar{g})$

We will first show that $x_f = x_g$ by contradiction. Suppose that $x_g \in \mathrm{supp}(\bar{f})$. If $\mathrm{supp}(\bar{g}) \subset \mathrm{supp}(\bar{f})$ then $\bar{f}$ witnesses that $\bar{f}$ and $\bar{g}$ cannot satisfy $\mathrm{SamePD}(\bar{f}, \bar{g})$. If $\mathrm{supp}(\bar{g}) \not\subset \mathrm{supp}(\bar{f})$ then the supports of $\bar{f}$ and $\bar{g}$ are as in the pictures in Case 2 of Figure 26.

Let $\bar{h}$ be a tuple such that:

1. $\mathrm{Aut}(M) \models \mathrm{Indec}(\bar{h})$;

2. $\{x_f\} = \text{Path}\langle \text{supp}(\bar{h}), M \setminus \text{supp}(\bar{h}) \rangle$; and

3. $\bar{f}$ and $\bar{g}$ act in different directions.

Then $\text{supp}(\bar{h}) \subset \text{supp}(\bar{g})$ and $\text{supp}(\bar{h}) \cap \text{supp}(\bar{f}) = \emptyset$, giving a contradiction.

Now suppose that $x_g \notin \text{supp}(\bar{f})$ and $x_f \notin \text{supp}(\bar{g})$. We consider two situations, where the point of $\text{Path}\langle x_f, x_g \rangle$ next to x_f is in the same direction as $\bar{f}$ or in the other direction (depicted in Figure 27).
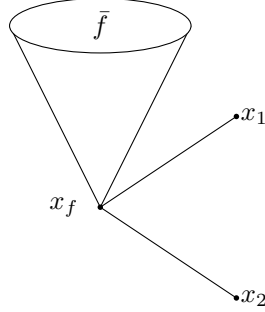

Figure 9: $\text{Path}\langle f, g \rangle$ and the Direction of $\bar{f}$

This picture depicts both situations. By "the point of $\text{Path}\langle x_f, x_g \rangle$ immediate to f is in the same direction as $\bar{f}$" we mean that $x_1 \in \text{Path}\langle f, g \rangle$, while $x_2 \in \text{Path}\langle f, g \rangle$ is the other situation we need to consider.

Suppose $x_1 \in \text{Path}\langle f, g \rangle$ and let ϕ be an automorphism of M which fixes f and switches x_1 with a member of $\text{supp}(\bar{f})$. Then $\phi * \bar{f}$ witnesses the fact that $\bar{f}$ and $\bar{g}$ cannot satisfy $\text{SamePD}(\bar{f}, \bar{g})$. If $x_2 \in \text{Path}\langle f, g \rangle$ then any tuple that satisfies Indec, fixes f and moves x_2 will do as a witness.

We know that if $\text{Aut}(M) \models \text{SamePD}(\bar{f}, \bar{g})$ then $x_f = x_g$. If $\bar{f}$ and $\bar{g}$ act in different directions then we may pick any point in $\text{supp}(\bar{g})$ and any tuple that fixes that point and moves x_f to find our counter-example.

It remains to show that if $\bar{f}$ and $\bar{g}$ fix the same point and have the same direction then they satisfy SamePD. Assume without loss of generality that $\bar{f}$ and $\bar{g}$ act on the successors of x_f. Let $\bar{h}$ be any tuple such that

$$[\text{supp}(\bar{f}) \sqsubset \text{supp}(\bar{h})]$$

This means that $\bar{h}$ moves x_f and all its successors, and therefore $\text{supp}(\bar{g})$ contains the support of $\bar{g}$, and so $\bar{h}$ satisfies $[\text{supp}(\bar{f}) \sqsubset \text{supp}(\bar{h})]$. $\qquad\square$

Lemma 3.18. *Recall that* $\text{RepPoint}(\bar{f}_0, \bar{f}_1)$ *is the formula*

$$\text{disj}(\bar{f}_0, \bar{f}_1) \wedge \forall \bar{g} \exists \bar{h}(\neg \text{disj}(\bar{g}, \bar{h}) \wedge (\text{SamePD}(\bar{f}_0, \bar{h}) \vee \text{SamePD}(\bar{f}_1, \bar{h})))$$

Let

$$\{x_0\} := \text{Path}\langle \text{supp}(\bar{f}_0), M \setminus \text{supp}(\bar{f}_0) \rangle$$
$$\{x_1\} := \text{Path}\langle \text{supp}(\bar{f}_1), M \setminus \text{supp}(\bar{f}_1) \rangle$$

Then

$$\text{Aut}(M) \models \text{RepPoint}(\bar{f}_0, \bar{f}_1)$$

if and only if $x_0 = x_1$ *and* $\bar{f}_0$ *and* $\bar{f}_1$ *act in different directions.*

Proof. First we will prove that if $\bar{f}_0$ and $\bar{f}_1$ are such that $x_0 = x_1$ and $\bar{f}_0$ and $\bar{f}_1$ act in different directions then

$$\text{Aut}(M) \models \text{RepPoint}(\bar{f}_0, \bar{f}_1)$$

If $\neg \text{disj}(\bar{g}, \bar{f}_0)$ or $\neg \text{disj}(\bar{g}, \bar{f}_1)$ then we may take $\bar{h} = \bar{f}_0$ or $\bar{h} = \bar{f}_1$, so suppose that $\text{disj}(\bar{g}, \bar{f}_0)$ and $\text{disj}(\bar{g}, \bar{f}_1)$.

Let

$$\{x_g\} := \text{Path}\langle \text{supp}(\bar{g}), M \setminus \text{supp}(\bar{g}) \rangle$$

and let $\bar{h}$ be such that

$$(\text{SamePD}(\bar{f}_0, \bar{h}) \vee \text{SamePD}(\bar{f}_1, \bar{h}))$$

and $\text{Path}\langle x_0, x_g \rangle \subset \text{supp}(\bar{h})$. Clearly this $\bar{h}$ is as required by the formula.

11

Now we must prove that if
$$\mathrm{Aut}(M) \models \mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$$
then $\bar{f}_0$ and $\bar{f}_1$ are as desired. If $x_0 \neq x_1$ then there is some y such that none of the following hold
$$y \in \mathrm{Path}\langle x_0, x_1\rangle \quad x_0 \in \mathrm{Path}\langle y, x_1\rangle \quad x_1 \in \mathrm{Path}\langle y, x_0\rangle$$
Let $\bar{g}$ be such that $\mathrm{Path}\langle y, \{x_0, x_1\}\rangle \not\subset \mathrm{supp}(\bar{g})$ and
$$y = \mathrm{Path}\langle \mathrm{supp}(\bar{g}), M \setminus \mathrm{supp}(\bar{g})\rangle$$
This $\bar{g}$ witnesses the fact that $\bar{f}_0$ and $\bar{f}_1$ do not satisfy $\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1)$.

Now suppose that $x_0 = x_1$ but
$$\mathrm{Aut}(M) \models \mathrm{SamePD}(\bar{f}_0, \bar{f}_1)$$
In this case any $\bar{g}$ whose support is disjoint from that of $\bar{f}_0$ and $\bar{f}_1$ and which fixes f_0 will be a witness. $\square$

We now have our formula that defines the domain of interpretation, however there will be a lot of pairs that satisfy RepPoint but fix the same point.

Lemma 3.19. *Recall that* $\mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$ *is the formula*

$$\mathrm{RepPoint}(\bar{f}_0, \bar{f}_1) \wedge \mathrm{RepPoint}(\bar{g}_0, \bar{g}_1)\wedge$$
$$(\mathrm{SamePD}(\bar{f}_0, \bar{g}_0) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_1)) \vee (\mathrm{SamePD}(\bar{f}_0, \bar{g}_1) \wedge \mathrm{SamePD}(\bar{f}_1, \bar{g}_0))$$

Let

$$x_f := \mathrm{Path}\langle \mathrm{supp}(\bar{f}_0), M) \setminus \mathrm{supp}(\bar{f}_0)\rangle$$
$$x_g := \mathrm{Path}\langle \mathrm{supp}(\bar{g}_0), M) \setminus \mathrm{supp}(\bar{g}_0)\rangle$$

If

$$\mathrm{Aut}(M) \models \mathrm{EqRepPoint}(\bar{f}_0, \bar{f}_1; \bar{g}_0, \bar{g}_1)$$

then $x_f = x_g$.

Proof. Clearly $x_f \neq x_g$ if and only if $\mathrm{SamePD}(\bar{f}_i, \bar{g}_j)$ holds for some choice of indices. $\square$

3.2 Interpreting Betweenness

From now on we will adopt the convention that when a lower case letter, such as g, appears in one of our formulas, it is actually a pair $(\bar{g}_0, \bar{g}_1)$ that satisfies RepPoint. We will refer to the point represented by g as x_g.

Definition 3.20. $\mathrm{Temp1PB}(g; h, k)$ *is the following formula:*

$$\exists l(\mathrm{EqRepPoint}(\bar{g}_0, \bar{g}_1; \bar{l}_0, \bar{l}_1) \wedge \left(\begin{array}{c} \neg\mathrm{disj}(\bar{l}_0, \bar{h}_0) \wedge \neg\mathrm{disj}(\bar{l}_0, \bar{h}_1)\wedge \\ \neg\mathrm{disj}(\bar{l}_1, \bar{k}_0) \wedge \neg\mathrm{disj}(\bar{l}_1, \bar{k}_1) \end{array} \right)$$
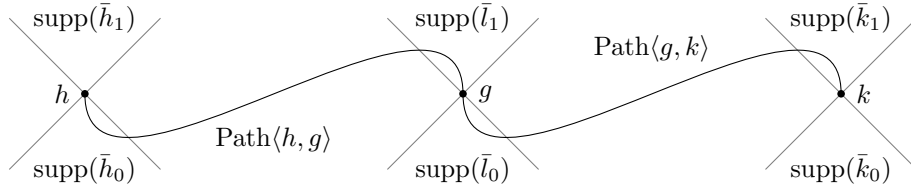

Figure 10: What is described by $\mathrm{Temp1PB}(g; h, k)$

$\mathrm{Temp2PB}(g; h, k)$ *is the formula*

$$\phi(g; h, k) \wedge \forall l \, \phi(l; h, k) \rightarrow \left[\begin{array}{cc} \mathrm{Temp1PB}(g; l, k) & \wedge \\ \mathrm{Temp1PB}(g; l, h) & \end{array} \right]$$
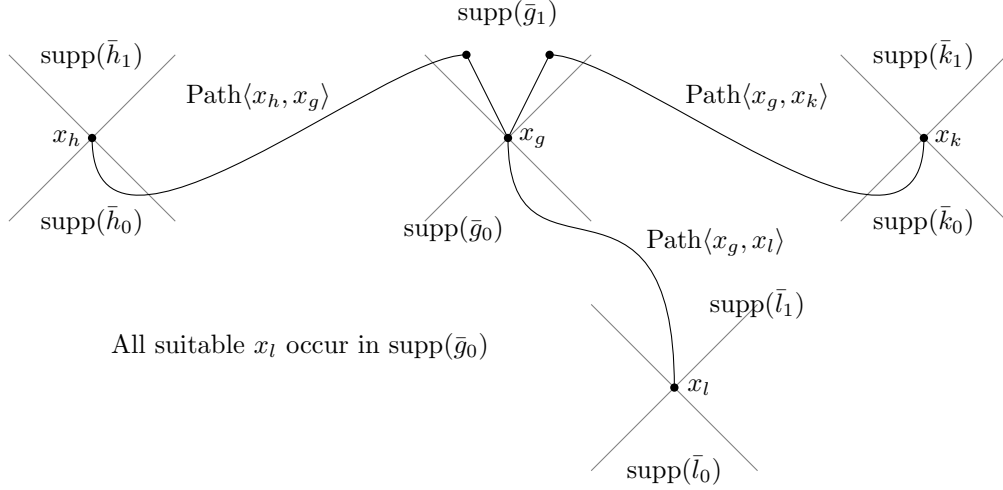
Figure 11: What is described by Temp2PB(g; h, k)

where ϕ is the formula that requires, using disj, the configurations of the supports of $\bar{g}_0$, $\bar{g}_1$, $\bar{h}_0$, $\bar{h}_1$, $\bar{k}_0$ and $\bar{k}_1$ depicted in Figure 29, for all permutations of the indices and that each pair represents different points.

PathBetween(g; h, k) is the formula

$$\text{Temp1PB}(g; h, k) \vee \text{Temp2PB}(g; h, k)$$

Lemma 3.21. *The previously defined formulas express the following properties of the structure:*

1. *Temp1PB(g; h, k) holds if and only if $\text{Path}\langle x_h, x_k \rangle$ contains a chain of length at least three, of which x_g is one of the middle points.*

2. *Temp2PB(g; h, k) holds only if x_g is a local maximum or minimum of $\text{Path}\langle x_h, x_k \rangle$.*

3. *PathBetween(x_g; x_h, x_k) holds if and only if $x_g \in \text{Path}\langle x_h, x_k \rangle$.*

Proof. Without loss of generality, we suppose that the situation is the same as depicted in the diagrams above.

1. Since the formula Temp1PB insists that $x_h \in \text{supp}(\bar{g}_1)$ and $x_k \in \text{supp}(\bar{g}_0)$, and since any path between something in $\text{supp}(\bar{g}_0)$ and something in $\text{supp}(\bar{g}_1)$ must pass through x_g, we conclude that $x_g \in \text{Path}\langle x_h, x_k \rangle$. Additionally, since $\bar{g}_0$ and $\bar{g}_1$ point in different directions there must be both an immediate successor and an immediate predecessor of x_g lying on $\text{Path}\langle x_h, x_k \rangle$ thus showing that if Temp1PB holds then the properties it was intended to describe hold. The other direction is immediate.

2. Since the formula Temp2PB holds both x_h and x_k are in $\text{supp}(\bar{g}_1)$, if $x_g \in \text{Path}\langle x_h, x_k \rangle$ then it is either a local maximum or a local minimum, as $\text{supp}(\bar{g}_0)$ is an extended connected component originating at x_g. If $x_g \notin \text{Path}\langle x_h, x_k \rangle$ then

$$\{x_g\} \subsetneq \text{Path}\langle x_g, x_h \rangle \cap \text{Path}\langle x_g, x_k \rangle$$

Any

$$x_l \in [\text{Path}\langle x_g, x_h \rangle \cap \text{Path}\langle x_g, x_k \rangle] \setminus \{x_g\}$$

will prevent Temp2PB from holding. Again, the other direction is immediate.

3. If $x_g \in \text{Path}\langle x_h, x_k \rangle$ then either x_g is a local maximum or minimum, or x_g lies on a chain of length at least 3, so Temp1PB and Temp2PB successfully cover every case.

$\square$

Definition 3.22. Related(f, g) *is the formula*

$$\forall h(\text{PathBetween}(h; f, g) \rightarrow \text{Temp1PB}(h; f, g))$$

13

Lemma 3.23. Related(f, g) *holds if and only if* $x_f \leq x_g$ *or* $x_g < x_f$.

At this point we have recovered M up to order reversal. We may, if we wish, recover the full order using a variety of different methods, which I will detail later, but from here we can prove that the class is faithful by recovering the betweenness reduct of the CFPOs in question.

Definition 3.24. $B(h; f, g)$ *is the formula*

$$\text{PathBetween}(h; f, g) \wedge \left(\begin{array}{cc} \text{Related}(f, g) & \wedge \\ \text{Related}(f, h) & \wedge \\ \text{Related}(g, h) & \end{array} \right)$$

Lemma 3.25. $B(h; f, g)$ *if and only if* x_h *is between* x_f *and* x_g.

Theorem 3.26. K_{Cone} *is faithful.*

Proof. Let $\langle M, \leq \rangle, \langle N, \leq \rangle \in K_{Cone}$. Let Φ be the first-order interpretation comprising of:

- RepPoint(x) as the formula that defines the domain of interpretation;
- EqRepPoint(x, y) as the equivalence relation on the domain of interpretation;
- $B(z; x, y)$ as the betweenness relation.

We have established previously that for all M

$$\Phi(\text{Aut}(\langle M, \leq \rangle)) \cong \langle M, B \rangle$$

Therefore Aut$\langle M, \leq \rangle \cong$ Aut$\langle N, \leq \rangle$ if and only if $\langle M, B \rangle \cong \langle N, B \rangle$.

If $\langle M, B \rangle \cong \langle N, B \rangle$ then $\langle M, \leq \rangle \cong \langle N, \leq \rangle$ or $\langle M, \leq \rangle \cong \langle N, \leq^* \rangle$ (the reverse ordering). By assumption, this means that $\langle M, \leq \rangle \cong \langle N, \leq \rangle$, thus the class is faithful. $\square$

4 Reconstructing the Order

It is impossible to reconstruct the order of all members of K_{Cone} with a first-order interpretation. Some members of K_{Cone} are isomorphic to their own reverse image, so the automorphism group has no way of knowing which way is 'up'.

In those circumstances, it will be necessary to make an artificial choice over which way is 'up'. When reconstructing linear orders in [1], McCleary and Rubin use a parameter pair for this purpose, obtaining a formula $\phi(x_1, x_2; y_1, y_2)$, which interprets

$$x_1 \leq x_2 \Leftrightarrow y_1 \leq y_2$$

This approach is also possible in this context, but not in a first order way.

Since all members of K_{Cone} embed the alternating chain, as the path between $\{x_1, x_2\}$ and $\{y_1, y_2\}$ grows, we require longer and longer formulas. We must use an $L_{\omega_1, \omega}$ formula to recover the order with this technique.

Another approach would be to exploit the fact that we have insisted that

$$ro \downarrow (M) \leq ro \uparrow (M)$$

Ramification order is definable when finite, so if $ro \downarrow (M) < \{ro \uparrow (M), \aleph_0\}$, then we can find a first order formula that depends on $ro \downarrow$ that interprets the order.

While first order, I find this far less satisfactory, as it gives lots of different formulas, each of which only work in limited circumstances. Even together they do not work everywhere. However, I will present both.

4.1 $ro \downarrow (M) < \{ro \uparrow (M), \aleph_0\}$

Definition 4.1. *Let* $K_{Cone}^n := \{M \in K_{Cone} : ro \downarrow (M) \leq n < ro \uparrow (M)\}$.

Definition 4.2. $x <_n y$ *is the following formula*

$$\text{Related}(x, y) \wedge \exists x_0, \ldots, x_n \left(\begin{array}{ll} \bigwedge_{0 \leq i \leq n} \text{Related}(x, x_i) & \wedge \\ \bigwedge_{i \neq j} \neg\text{Related}(x_i, x_j) & \wedge \\ \bigwedge_{i \neq j} \text{PathBetween}(x; x_i, x_j) & \wedge \\ \neg\text{PathBetween}(x; y, x_0) & \end{array} \right)$$

Theorem 4.3. *If $M \in K_{Cone}^n$ then $\mathrm{Aut}(M) \models x \lll_n y$ if and only if $M \models x <_M y$.*

Proof. By definition, $x \lll_n y \to \mathrm{Related}(x, y)$, so if $\mathrm{Aut}(M) \models x \lll_n y$ then $M \models x \leq\geq_M y$.

Each of the x_i are related to x, but $\{x_i : i = 0, ..., n\}$ forms an antichain. Suppose that none of the x_i's lie above x. Since $ro \downarrow (M) \leq n$ this means that at least two of the x_i's, say x_0 and x_1, are contained in the same downwards cone of x.

Therefore $x_0 \vee x_1 < x$, but the connecting set of the path from x_0 to x_1 must be

$$\{x_0, x_0 \vee x_1, x_1\}$$

which would imply that $x \notin \mathrm{Path}\langle x_0, x_1 \rangle$, which contradicts the assumption that $\mathrm{Aut}(M) \models x \lll_n y$. Thus at least one of the x_i's is above x.

Suppose, without loss of generality, that x_0 is above x. If any of the other x_i's lie below x_0 then they will be related to x_i, giving a contradiction. By the above argument, all of the x_i's lie in different cones.

On the other hand, any $n + 1$ element antichain above x, where every element is contained in a different cone above x satisfies the all of the properties demanded of it, except

$$\left(\bigvee_{i \leq n} \neg \mathrm{PathBetween}(x; y, x_i) \right)$$

If $x < y$ then we will be able to choose x_0 such that x_0 is contained in the same cone as y, so any such antichain will satisfy the formula.

If $y < x$ then any path from any of the x_i's to y will pass through x, and so the formula cannot be satisfied. $\qquad \square$

4.2   Abandoning First Order Logic

Throughout this subsection, we assume that y_1 and y_2 satisfy Related. All the formulas mentioned will use y_1 and y_2 as parameters. We will use y_1 and y_2 to indicate the direction of the order, so we suppose that $y_1 < y_2$.

Definition 4.4. *$(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$ is the formula that insists that x_1, x_2, y_1 and y_2 are all related and using $B(z; x, y)$ insists that they lie in one of the configurations depicted below.*
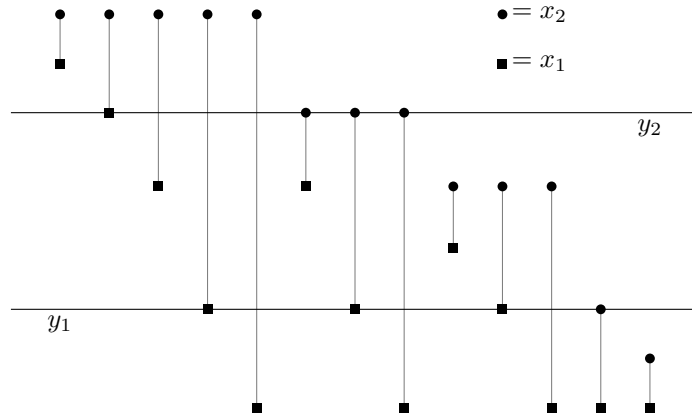

Figure 12: Defining $(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$

Lemma 4.5. *If $\mathrm{Aut}(M) \models (x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$ then $M \models x_1 <_M x_2$.*

Proof. All possible cases are covered by the definition. $\qquad \square$

Definition 4.6. *$(x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$ is the formula*

$$\neg(x_2 <_0 x_1 \Leftrightarrow y_1 < y_2) \wedge \neg(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2) \wedge (\alpha_1 \vee \alpha_2 \vee \alpha_3 \vee \alpha_4)$$

where:

$$\alpha_1 := \quad B(y_2; y_1, x_2) \wedge \text{Related}(x_1, x_2)$$
$$\alpha_2 := \quad B(x_2; x_1, y_2)$$
$$\alpha_3 := \quad B(y_1; x_1, y_2) \wedge \text{Related}(x_1, x_2)$$
$$\alpha_4 := \quad B(x_1; y_1, x_2)$$
$$\alpha_5 := \quad \bigwedge_{i \neq j} \text{Related}(x_i, y_j) \wedge \bigwedge_{i=j} \neg\text{Related}(x_i, y_j) \wedge \text{Related}(x_1, x_2)$$
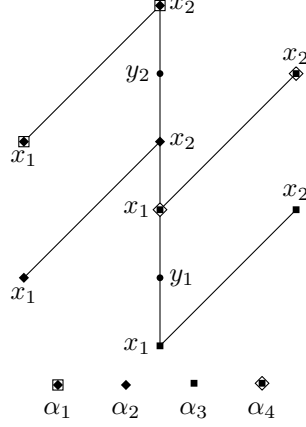

Figure 13: Defining $(x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$

Lemma 4.7. *If* $\text{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$ *then* $M \models x_1 <_M x_2$.

Proof. Let $(x_0, x_1) \in M$ be such that $\text{Aut}(M) \models \neg(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$. We will show that when $\text{Aut}(M) \models \alpha_i$ then $x_1 <_M x_2$ for each possible i.

First, assume that $\text{Aut}(M) \models \alpha_1$. Since $\text{Aut}(M) \models B(y_2; y_1, x_2)$ and we are supposing that $y_1 <_M y_2$, we know that $x_2 >_M y_2$. We also know that x_1 cannot be greater than x_2, as otherwise $\text{Aut}(M) \models (x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$. Since we have asserted that $\text{Aut}(M) \models \text{Related}(x_1, x_2)$, this means that $x_1 <_M x_2$.

Now we assume that $\text{Aut}(M) \models \alpha_2$, so either $x_1 <_M x_2 <_M y_2$ or $y_2 <_M x_2 <_M x_1$, but the latter contradicts our assertion that not both of x_1 and x_2 are related to both y_1 and y_2.

Assume that $\text{Aut}(M) \models \alpha_3$, so $x_1 <_M y_1 <_M y_2$. If $x_2 <_M x_1$ then $x_2 <_M y_1, y_2$, contradicting $\text{Aut}(M) \models \neg(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$.

Assume that $\text{Aut}(M) \models \alpha_2$, so either $x_2 <_M x_1 <_M y_1$ or $y_1 <_M x_1 <_M x_2$, but the former contradicts our assertion that not both of x_1 and x_2 are related to both y_1 and y_2.

Assume that $\text{Aut}(M) \models \alpha_5$, so $\text{Aut}(M) \models \text{Related}(x_1, y_2) \wedge \neg\text{Related}(x_1, y_1)$. This means that $x_1 <_M y_2$. If $x_2 <_M x_1$ then $x_2 <_M y_2$, but we have asserted that $\text{Aut}(M) \models \neg\text{Related}(x_2, y_2)$. $\square$

Definition 4.8. *Let* $n \geq 2$. *The formula* $(x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ *is defined to be the conjunction of the following four formulas:*

$$\forall z \left(\bigwedge_{i < n-1} \neg(x_1 <_i z \Leftrightarrow y_1 < y_2) \right) \wedge \forall z \left(\bigwedge_{i < n-1} \neg(z <_i x_2 \Leftrightarrow y_1 < y_2) \right)$$

to ensure that the order is yet to be resolved for either x_1 *or* x_2;

$$(\exists z \, ((x_1 <_{n-1} z \Leftrightarrow y_1 < y_2))) \wedge \forall z \, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2)))$$
$$\vee$$
$$(\exists z \, ((z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))) \wedge \forall z \, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2)))$$

to ensure that exactly one of x_1 *and* x_2 *is related by* $<_{n-1}$ *to something;*

$$\forall z \, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2)) \rightarrow$$
$$\exists w \, ((x_1 <_{n-1} w \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow x_1 < w))$$

to describe what happens when x_1 *is in the area where the order is defined, but* x_2 *is not, and;*

$$\forall z \, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2)) \rightarrow$$
$$\exists w \, ((w <_{n-1} x_2 \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow w < x_2))$$

16

to describe what happens when x_2 is in the area where the order is defined, but x_1 is not.

Proposition 4.9. *If* $\mathrm{Aut}(M) \models (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ *then* $M \models x_1 <_M x_2$.

Proof. We proceed by induction, starting with the base case $n = 2$. Suppose that

$$\mathrm{Aut}(M) \models (x_1 <_2 x_2 \Leftrightarrow y_1 < y_2)$$

If

$$\mathrm{Aut}(M) \models \forall z \, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))$$

then there is a $w \in M$ such that $\mathrm{Aut}(M) \models (x_1 <_1 w \Leftrightarrow y_1 < y_2)$ and so $x_1 <_M w$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow x_1 < w)$ implies that $x_1 <_M x_2$.

If $\mathrm{Aut}(M) \models \forall z \, (\neg(x_1 <_1 z \Leftrightarrow y_1 < y_2))$ then there is a $w \in M$ such that

$$\mathrm{Aut}(M) \models (w <_1 x_2 \Leftrightarrow y_1 < y_2)$$

and so $w <_M x_2$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow w < x_2)$ implies that $x_1 <_M x_2$.

We now examine the induction step. Suppose that if $\mathrm{Aut}(M) \models (x_1 <_{n-1} x_2 \Leftrightarrow y_1 < y_2)$ then $x_1 <_M x_2$ and also suppose that $\mathrm{Aut}(M) \models (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$.

If $\mathrm{Aut}(M) \models \forall z \, (\neg(z <_{n-1} x_2 \Leftrightarrow y_1 < y_2))$ then there is a $w \in M$ such that

$$\mathrm{Aut}(M) \models (x_1 <_{n-1} w \Leftrightarrow y_1 < y_2)$$

and so $x_1 <_M w$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow x_1 < w)$ implies that $x_1 <_M x_2$.

If $\mathrm{Aut}(M) \models \forall z \, (\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2))$ then there is a $w \in M$ such that

$$\mathrm{Aut}(M) \models (w <_{n-1} x_2 \Leftrightarrow y_1 < y_2)$$

and so $w <_M x_2$. Therefore $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow w < x_2)$ implies that $x_1 <_M x_2$. $\qquad\square$

Definition 4.10. $(x_1 < x_2 \Leftrightarrow y_1 < y_2)$ *is defined to be the* $L_{\omega_1,\omega}$*-formula:*

$$\bigvee_{n<\omega} (x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$$

Theorem 4.11. $\mathrm{Aut}(M) \models (x_1 < x_2 \Leftrightarrow y_1 < y_2)$ *if and only if* $M \models x_1 <_M x_2$.

Proof. Suppose $\mathrm{Aut}(M) \models (x_1 < x_2 \Leftrightarrow y_1 < y_2)$. In the first clause of their definition, we ensured that each of the formulas $(x_1 <_n x_2 \Leftrightarrow y_1 < y_2)$ are mutually exclusive. By Lemmas 4.5 and 4.7 and Proposition 4.9, so no matter which $\mathrm{Aut}(M)$ realises, we have ensured that $x_1 <_M x_2$.

Suppose $M \models x_1 <_M x_2$. We examine the length of $\mathrm{Path}\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$, which we shall call j. If $j \leq 2$ then at least one of x_1 and x_2 is related to at least one of y_1 and y_2.

Suppose that both x_1 and x_2 are related to both y_1 and y_2. Since x_1 and x_2 must occur in one of the situations described by $(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2)$.

Now suppose that not both of x_1 and x_2 are related to both y_1 and y_2, but at least one is. This situation is fully described by $(x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$.

If neither x_1 nor x_2 are related to either y_1 or y_2 then $\mathrm{Aut}(M)$ realises neither

$$(x_1 <_0 x_2 \Leftrightarrow y_1 < y_2) \quad \text{nor} \quad (x_1 <_1 x_2 \Leftrightarrow y_1 < y_2)$$

as both of those formulas contain instances of $B(z; x, y)$ that prevent this.

Now suppose that $j \geq 3$. We also assume that for all z_1 and z_2 such that $z_1 <_M z_2$, the length of $\mathrm{Path}\langle\{z_1, z_2\}, \{y_1, y_2\}\rangle$ is i for $i < j$ if and only if

$$\mathrm{Aut}(M) \models (z_1 <_i z_2 \Leftrightarrow y_1 < y_2)$$

Suppose $\mathrm{Path}\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle = j$. We first wish to show that (x_0, x_1) satisfies the first clause of $(x_1 < x_2 \Leftrightarrow y_1 < y_2)$, vis.

$$\forall z \left(\bigwedge_{i<j-1} \neg(x_1 <_i z \Leftrightarrow y_1 < y_2) \right) \wedge \forall z \left(\bigwedge_{i<n-1} \neg(z <_i x_2 \Leftrightarrow y_1 < y_2) \right)$$

If there is a z such that $(x_1 <_i z \Leftrightarrow y_1 < y_2)$ for some $i < j - 1$ then, by the induction hypothesis, the length of $\mathrm{Path}\langle\{z, x_1\}, \{y_1, y_2\}\rangle$ is less than $j - 1$. Since $x_1 <_M x_2$, this means that the length of $\mathrm{Path}\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$ is less than j, contradicting our assumptions. If there is a z such that $(z <_i x_2 \Leftrightarrow y_1 < y_2)$ then we reach a similar contradiction.

Let us now examine the second clause:

$$\forall z\,(\neg(z <_{j-1} x_2 \Leftrightarrow y_1 < y_2)) \to$$
$$\exists w\,((x_1 <_{j-1} w \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow x_1 < w))$$

If there is a z such that $\mathrm{Aut}(M) \models (z <_{j-1} x_2 \Leftrightarrow y_1 < y_2)$ then we are done, so suppose that there is no such z. Let $z_1, \ldots, z_j$ be the elements of the connecting set of $\mathrm{Path}\langle \{x_1, x_2\}, \{y_1, y_2\}\rangle$, such that z_1 is related to x_1 and x_2.

If $z_2 <_M z_1 \leq_M x_2$ then $\mathrm{Aut}(M) \models (z_2 <_{j-1} x_2 \Leftrightarrow y_1 < y_2)$, so we may assume that $x_1 \leq_M z_1 <_M z_2$.

$\mathrm{Path}\langle \{z_1, z_2\}, \{y_1, y_2\}\rangle = \mathrm{Path}\langle z_2, \{y_1, y_2\}\rangle$ has length $j - 1$, so $\mathrm{Aut}(M) \models (x_1 <_{j-1} z_2 \Leftrightarrow y_1 < y_2)$. Additionally, we have deduced that $z_2 \parallel x_2$, and $x_1 \leq z_2, x_2$, so $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow x_1 < z_2)$, so (x_1, x_2) satisfies the second clause.

Recall that the third clause we must examine is:

$$\forall z\,(\neg(x_1 <_{n-1} z \Leftrightarrow y_1 < y_2)) \to$$
$$\exists w\,((w <_{n-1} x_2 \Leftrightarrow y_1 < y_2) \wedge (x_1 <_1 x_2 \Leftrightarrow w < x_2))$$

If there is a z such that $\mathrm{Aut}(M) \models (x_1 <_{j-1} z \Leftrightarrow y_1 < y_2)$ then we are done, so suppose that there is no such z. Again, let $z_1, \ldots, z_j$ be the elements of the connecting set of $\mathrm{Path}\langle \{x_1, x_2\}, \{y_1, y_2\}\rangle$, such that z_1 is related to x_1 and x_2.

If $x_1 \leq_M z_1 <_M z_2$ then $\mathrm{Aut}(M) \models (x_1 <_{j-1} z_2 \Leftrightarrow y_1 < y_2)$, so we may assume that $z_2 <_M z_1 \leq_M x_2$.

$\mathrm{Path}\langle \{z_1, z_2\}, \{y_1, y_2\}\rangle = \mathrm{Path}\langle z_2, \{y_1, y_2\}\rangle$ has length $j - 1$, so

$$\mathrm{Aut}(M) \models (z_2 <_{j-1} x_2 \Leftrightarrow y_1 < y_2)$$

Additionally, we have deduced that $z_2 \parallel x_1$, and $x_2 \geq z_2, x_1$, so $\mathrm{Aut}(M) \models (x_1 <_1 x_2 \Leftrightarrow z_2 < x_1)$, so (x_1, x_2) satisfies the third clause.

Now suppose that $\mathrm{Aut}(M) \models (x_1 <_j x_2 \Leftrightarrow y_1 < y_2)$. Let $z_k, \ldots, z_j$ be the elements of the connecting set of $\mathrm{Path}\langle \{x_1, x_2\}, \{y_1, y_2\}\rangle$, such that z_k is related to x_1 and x_2. Since

$$\mathrm{Aut}(M) \models \forall z \left(\bigwedge_{i<j-1} \neg(x_1 <_i z \Leftrightarrow y_1 < y_2) \right) \wedge \forall z \left(\bigwedge_{i<n-1} \neg(z <_i x_2 \Leftrightarrow y_1 < y_2) \right)$$

the length of $\mathrm{Path}\langle \{x_1, x_2\}, \{y_1, y_2\}\rangle$ has at least j elements, and thus $k \leq 1$. By the induction hypothesis, either

$$\mathrm{Aut}(M) \models (z_1 <_{j-1} z_2 \Leftrightarrow y_1 < y_2) \quad \text{or} \quad \mathrm{Aut}(M) \models (z_2 <_{j-1} z_1 \Leftrightarrow y_1 < y_2)$$

so if $k \neq 1$ then (x_1, x_2) cannot possibly satisfy the second and third coordinates. $\square$

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